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Homoclinic orbits of positive definite Lagrangian systems

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Abstract

If the Aubry set $\widetilde{\mathcal{A}}(c)$ satisfies some topological hypothesis, such as $H_1(M \times \mathbb{T}, \mathcal{A}(c), \mathbb{R}) \neq 0$, then the α-function has a flat. In this paper, we will prove that $\widetilde{\mathcal{A}}(c')$ has infinitely many $\widetilde{M}$-minimal homoclinic orbits when c' is on the boundary of the maximal flat of the α-function. These homoclinic orbits are different from the usually called multi-bump solutions.

Keywords: Positive definiteness; Lagrangian system; Homoclinic orbit; Aubry set

1. Introduction and main results

Let M be a compact, connected and C^∞ manifold, TM its tangent bundle. We assume that the Lagrangian $L : TM \times \mathbb{R} \to \mathbb{R}$ is a C^2 function and satisfies the following conditions

Periodicity. The Lagrangian L is 1-periodic with respect to time, i.e., $L(z,t) = L(z,t+1)$ for all $z \in TM$ and $t \in \mathbb{R}$.

Positive definiteness. For each $m \in M$ and $t \in \mathbb{R}$, the restriction $L|_{TM_m \times t}$ has everywhere positive definite Hessian second derivative. If we let $x = (x_1, x_2, \ldots, x_n)$ be a local coordinates of an open neighborhood $U \subset M$ of m, then $(x, \dot{x}) = (x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$ is a local canonical coordinates of $\pi^{-1}U$, where π is the projection along the tangent fibers. This condition means that $L_{\dot{x}\dot{x}}$ is a positive definite matrix.

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Superlinear growth. We suppose that L has fiberwise superlinear growth:

$$\frac{L(\xi, t)}{\|\xi\|} \to +\infty, \quad \text{as } \|\xi\| \to +\infty, \text{ for } t \in \mathbb{R}.$$

Here, $\| \ \|$ denotes the norm associated to a Riemannian metric on M. Obviously, this condition is independent of Riemannian metric since M is compact.

Completeness. All solutions of the Lagrange equations are well defined for all $t \in \mathbb{R}$.

Let $I = [a, b]$ be a compact interval of time. A curve $\gamma \in C^1(I, M)$ is called a c-minimizer or c-minimal curve if it minimizes the action functional among all curves $\xi \in C^1(I, M)$ which satisfy the same boundary conditions:

$$A_c(\gamma) = \min_{\substack{\xi(a)=\gamma(a) \\ \xi(b)=\gamma(b)}} \int_a^b (L - \eta_c)\big(d\xi(t), t\big)\, dt.$$

If J is not a compact interval, the curve $\gamma \in C^1(J, M)$ is called a c-minimizer if $\gamma|_I$ is c-minimal for any compact interval $I \subset J$. An orbit $X(t)$ of the Euler–Lagrange flow Φ^t is called c-minimizing if the curve $\pi \circ X$ is c-minimizer. A point $(z, s) \in TM \times \mathbb{T}$ is c-minimizing if its orbit $X(t)$ is c-minimizing. Denoting all c-minimizing points by $\widetilde{\mathcal{G}}$, we know that it is a nonempty compact subset of $TM \times \mathbb{T}$, invariant for the Euler–Lagrange flow Φ^t [2].

Let $\mathcal{M}$ be the set of Φ^t-invariant probability measures on $TM \times \mathbb{T}$. For each measure $\nu \in \mathcal{M}$, we can define the action $A_c(\nu)$ as follows:

$$A_c(\nu) = \int (L - \eta_c)\, d\nu,$$

where η_c is a closed 1-form on M whose de Rham cohomology class is c, i.e., $[\eta_c] = c$. As showed in [11], there is a measure $\nu \in \mathcal{M}$ such that A_c attains its minimum. We use $-\alpha(c)$ to denote this minimum and call this measure c-minimal invariant measure. The union of the supports of all c-minimal invariant measures is the Mather set, denoted by $\widetilde{\mathcal{M}}(c)$. It is easy to verify that $\alpha(c)$ is a convex, finite everywhere and superlinear growth function on $H^1(M, \mathbb{R})$, usually called α-function. Let $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$ denote the conjugate function of $\alpha(c)$ in the sense of convex analysis [13], i.e.,

$$\beta(h) = \max_{c \in H^1(M,\mathbb{R})} \big\{\langle c, h\rangle - \alpha(c)\big\},$$

where $\langle\, ,\rangle$ denotes the canonical pairing between cohomology and homology. Obviously, β is also a convex, finite everywhere and superlinear growth function, usually called β-function.

To define Aubry set and Mañé set, we let

$$h_c\big((m, t), (m', t')\big) = \min_{\substack{\gamma \in C^1([t,t'], M) \\ \gamma(t)=m, \gamma(t')=m'}} \int_t^{t'} \big(L - \eta_c + \alpha(c)\big)\big(d\gamma(s), s\big)\, ds,$$

$$\Phi_c\big((m,s),(m',s')\big)=\inf_{\substack{t=s \bmod 1,\, t'=s' \bmod 1\\ t'-t\geqslant 1}} h_c\big((m,t),(m',t')\big),$$

$$h_c^\infty\big((m,s),(m',s')\big)=\liminf_{\substack{t=s \bmod 1,\, t'=s' \bmod 1\\ t'-t\to+\infty}} h_c\big((m,t),(m',t')\big),$$

$$h_c^k(m,m')=h_c\big((m,0),(m',k)\big),\qquad \Phi_c(m,m')=\Phi_c\big((m,0),(m',0)\big),$$

$$h_c^\infty(m,m')=h_c^\infty\big((m,0),(m',0)\big),\qquad d_c(m,m')=h_c^\infty(m,m')+h_c^\infty(m',m).$$

It was showed in [12] that d_c is a pseudo-metric on the set $\{x\in M\colon\ h_c^\infty(x,x)=0\}$. A curve $\gamma\in C^1(\mathbb{R},M)$ is called c-semi-static if

$$A_c(\gamma|_{[a,b]})+\alpha(c)(b-a)=\Phi_c\big(\big(\gamma(a),a \bmod 1\big),\big(\gamma(b),b \bmod 1\big)\big)$$

for each $[a,b]\subset\mathbb{R}$. A curve $\gamma\in C^1(\mathbb{R},M)$ is called c-static if

$$A_c(\gamma|_{[a,b]})+\alpha(c)(b-a)=-\Phi_c\big(\big(\gamma(b),b \bmod 1\big),\big(\gamma(a),a \bmod 1\big)\big)$$

for each $[a,b]\subset\mathbb{R}$. An orbit $X(t)=(d\gamma(t),t \bmod 1)$ is called c-static (semi-static) if γ is c-static (semi-static). A c-static curve is also c-semi-static.

We call the Mañé set $\widetilde{\mathcal{N}}(c)$ the union of all global c-semi-static orbits and the Aubry set $\widetilde{\mathcal{A}}(c)$ the union of all global c-static orbits. Both the Aubry set and the Mañé set have different names in [12]. These two sets can also be defined for some covering manifold $\widetilde{M}$ of M. Obviously, the c-static (semi-static) orbits for $\widetilde{M}$ are not necessarily c-static (semi-static) for M.

Let us denote the standard projection of $\widetilde{\mathcal{G}}(c)$, $\widetilde{\mathcal{N}}(c)$, $\widetilde{\mathcal{A}}(c)$ and $\widetilde{\mathcal{M}}(c)$ from $TM\times\mathbb{T}$ to $M\times\mathbb{T}$ by $\mathcal{G}(c)$, $\mathcal{N}(c)$, $\mathcal{A}(c)$ and $\mathcal{M}(c)$, respectively. We have the following inclusions [2]:

$$\widetilde{\mathcal{G}}(c)\supseteq\widetilde{\mathcal{N}}(c)\supseteq\widetilde{\mathcal{A}}(c)\supseteq\widetilde{\mathcal{M}}(c).$$

To present our main result, we need the following hypotheses:

(H_1) For a given c, the c-minimal measure is uniquely ergodic.

Without losing of generality, we can suppose $c=0$ and $\alpha(c)=0$. In fact, if $c\neq 0$ and $\alpha(c)\neq 0$, let us consider a new Lagrangian $\widetilde{L}=L-\eta_c+\alpha(c)$, where η_c is a closed 1-form whose first cohomology class is c, i.e., $[\eta_c]=c$. By some simple computation, the α-function of this new Lagrangian $\widetilde{L}$ has the property we expected.

The second hypothesis is:

(H_2) $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})\neq 0$.

The group $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$ is the Cěch homology group [14], defined as the inverse limit of $H_1(M\times\mathbb{T},U,\mathbb{R})$: $\lim_{\mathcal{A}(0)\subset U} H_1(M\times\mathbb{T},U,\mathbb{R})$, where $U\subset M\times\mathbb{T}$ is an open neighborhood of $\mathcal{A}(0)$. Since $\widetilde{\mathcal{M}}(0)$ is uniquely ergodic, $\widetilde{\mathcal{A}}(0)$ is composed of only one static class, consequently, it is a connected subset of $TM\times\mathbb{T}$ [5]. By the Lipschitz graph property of Aubry set, $\mathcal{A}(0)$ is a connected subset of $M\times\mathbb{T}$. So we can take a connected open ϵ-neighborhood U of $\mathcal{A}(0)$ in $M\times\mathbb{T}$.

Let $i:U\to M\times\mathbb{T}$ be the inclusion map, then

$$H_1(M\times\mathbb{T},U,\mathbb{R})\simeq H_1(M\times\mathbb{T},\mathbb{R})/i_*\big(H_1(U,\mathbb{R})\big).$$

Let $\widetilde{M}$ be the covering space of M defined by

$$\pi_1(\widetilde{M})=\ker\big(\mathcal{H}:\pi_1(M)\to H_1(M,\mathbb{R})\big),$$

where $\mathcal{H}$ is the Hurewicz map. The Deck transformation group of this covering is

$$H_1(M,\mathbb{Z})=\operatorname{Im}\big(\mathcal{H}:\pi_1(M)\to H_1(M,\mathbb{R})\big).$$

From this covering, we can induce a natural covering space of $M\times\mathbb{T}$ and denote by $\widetilde{M}\times\mathbb{R}$. The corresponding Deck transformation group is

$$H_1(M\times\mathbb{T},\mathbb{Z})=H_1(M,\mathbb{Z})\times\mathbb{Z}.$$

We define $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})$ as follows:

$$H_1\big(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z}\big)=H_1(M\times\mathbb{T},\mathbb{Z})/i_*\big(H_1(U,\mathbb{Z})\big),$$

for sufficiently small neighborhood U of $\mathcal{A}(0)$. Obviously, $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})$ is a nontrivial free Abel group and if we choose U small enough, we have $\operatorname{rank}H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})=\operatorname{rank}H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$. Our hypothesis ($H_2$) means that $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})$ is nonzero.

Remark 1.1. Since the support of the 0-minimal measure is contained in $\widetilde{\mathcal{A}}(0)$, its rotation vector ρ is the zero element of $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$, i.e., $\rho=0\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$.

Under the hypothesis (H_2), the α-function has a flat $P\subset H^1(M,\mathbb{R})$ whose dimension is at least $\operatorname{rank}H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$.

In fact, let c be an element of $H^1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$ which is dual to $H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$. Then there is a closed 1-form ν_c satisfying $[\nu_c]=c$ and $\operatorname{supp}\nu_c\cap U=\emptyset$. By the upper semi-continuity of set value function $c\to\widetilde{\mathcal{N}}(c)$, $\mathcal{N}(\lambda c)$ is in U and so is $\mathcal{M}(\lambda c)$ for small λ. Since $\nu_c|_U=0$, $\widetilde{\mathcal{M}}(\lambda c)=\widetilde{\mathcal{M}}(0)$. If we suppose the 0-minimal invariant measure is μ_0, then

$$-\alpha(\lambda c)=\int(L-\lambda\nu_c)\,d\mu_0=\int L\,d\mu_0-\big\langle\lambda c,\rho(\mu_0)\big\rangle=-\alpha(0),$$

where the last equality follows from the fact that $\nu_c|_U=0$ and $\operatorname{supp}\mu_0\subset\pi^{-1}U$.

By the convexity of α-function, there exists a small constant $\lambda>0$ such that $\alpha(c')=\alpha(0)$ for all $c'\in[-\lambda c,\lambda c]$. This means that $[-\lambda c,\lambda c]\subset P$. So the dimension of the flat P is at least $\operatorname{rank}H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R})$.

Obviously, a flat of α-function is a closed convex subset of $H^1(M,\mathbb{R})$. Let P be the maximal flat containing 0 as its interior point and P_0 be a subset of P defined by the following formula:

$$P_0=P\cap H^1\big(M\times\mathbb{T},\mathcal{A}(0),\mathbb{R}\big).$$

According to the argument above, $0\in\operatorname{int}P_0$ and $\alpha(c)=0$ for all $c\in P_0$.

The Aubry set $\widetilde{\mathcal{A}}(c)$ remains unchanged when we move c in the interior of P, i.e., $\widetilde{\mathcal{A}}(c) = \widetilde{\mathcal{A}}(c')$ if $c, c' \in \operatorname{int} P$. It may become larger when the first cohomology class is on the boundary of P. Let us consider $c \in \partial P_0$ from now on.

For any $0 \neq g \in H_1(M \times \mathbb{T}, \mathcal{A}_0, \mathbb{Z})$, we define

$$h_c^n(g) = \min_{\substack{\gamma(0)=\gamma(n)=x\in\mathcal{M}(0)|_{t=0}\\ [\gamma|_{[0,n]}]=g}} \int_0^n (L-\eta_c)\big(d\gamma(s), s\big)\,ds, \qquad h_c^\infty(g) = \liminf_{n\to+\infty} h_c^n(g),$$

where $[\eta_c] = c$ and $\operatorname{supp}\eta_c \cap U = \emptyset$.

Obviously, these definitions are independent of the choice of $x \in \mathcal{M}(0)|_{t=0}$, since $\widetilde{\mathcal{M}}(0)$ is uniquely ergodic. Obviously, $h_c^\infty(g) \geqslant 0$ for all $g \in H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{Z})$ since $x \in \mathcal{M}(c)$.

The third hypothesis is:

(H_3) For some $c \in \partial P_0$, there is a positive number δ such that

$$\inf_{g\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})} h_c^\infty(g) = \delta > 0.$$

Remark 1.2. If $\inf_{g\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})} h_c^\infty(g) = 0$, then there are two possibilities:

1. There is a $g \in H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{Z})$ such that $h_c^\infty(g) = 0$ In this case, there exist some homoclinic orbits of $\widetilde{\mathcal{A}}(0)$, which is in the Aubry set $\widetilde{\mathcal{A}}(c)$. There might be no other c-minimal measure other than μ_0 when c is on the boundary of P. In this case, the c'-minimal measure has its support in a small neighborhood of $\widetilde{\mathcal{A}}(c)$ if c' is outside P but close to and there exist infinitely many multi-bump homoclinic orbits to $\widetilde{\mathcal{A}}(0)$.
2. There exists a sequence of $\{g_i\}_{i=1}^\infty$ such that $h_c^\infty(g_i) > 0$ but $\lim_{i\to\infty} h_c^\infty(g_i) = 0$. In this case, there is at least one new c-minimal measure when c is on the boundary of P. Since the Mather set is the graph of some Lipschitz function defined on underling manifold M and invariant for the Euler–Lagrange flow, the support of this new minimal measure cannot intersect to $\widetilde{\mathcal{A}}(0)$. So hypothesis ($H_3$) can be satisfied by small perturbation of the Lagrangian. In fact, we only need add some small positive smooth function defined on M to the Lagrangian, it is zero on the supports of the new minimal measure and $\mathcal{A}(0)$. Under this perturbation, c is still on the flat of the α-function. For this Lagrangian, hypothesis (H_3) is satisfied.

Under these three hypotheses, we claim:

Theorem 1.1. *There exist infinitely many $\widetilde{M}$-minimal homoclinic orbits to $\widetilde{\mathcal{A}}(0)$, where the $\widetilde{M}$-minimal means that the homoclinic orbits, as a curve in the covering space $\widetilde{M}$, are the Tonelli's minimizers* [11].

The problem of homoclinic orbits in positive definite Lagrangian (or Hamiltonian) systems has been researched by several authors before us. In [3], Bolotin showed the existence of homoclinic orbits to the hyperbolic tori for positive definite Lagrangians, and in [4], he established the existence of infinitely many homoclinic orbits to the lower-dimensional tori. These homoclinic orbit looks like multi-bump solution. When the Aubry set does not have manifold structure, the existence of infinitely many multi-bump homoclinic orbits is proved by Cui and his collaborators in [6]. Obviously, these multi-bump solutions are not c-minimal in the covering space of M.

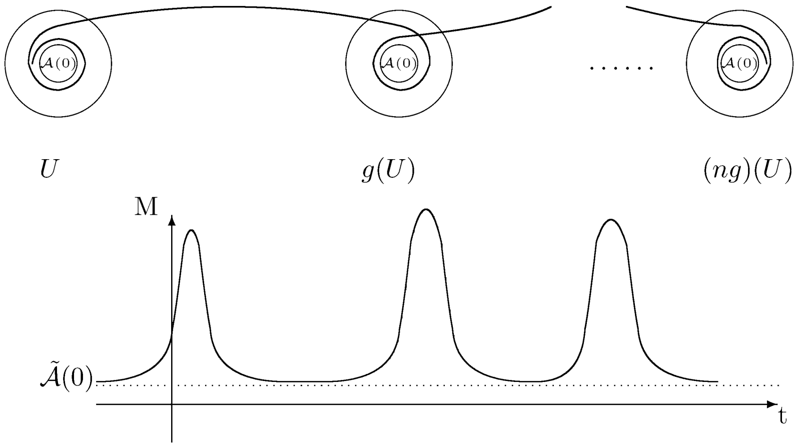

Fig. 1. Multi-bump homoclinic orbits.

Existence of $k+1$ homoclinic orbits to Aubry set is established by Bernard earlier in [1]. The existence of infinitely many homoclinic orbits and of positive entropy orbits under different kinds of hypotheses is obtained in [2]. In [7], the existence of homoclinic orbits is established and similar results are also obtained in [5].

In this paper we study an infinite sequence of homoclinic orbits which do not return to the small neighborhood of $\widetilde{\mathcal{A}}(0)$ for many times. Actually, they are c-minimal and stay away from a neighborhood of $\widetilde{\mathcal{A}}(0)$ for longer and longer period. It implies these homoclinic orbits approach to at least one c-minimal measure whose support is not contained in a neighborhood of $\widetilde{\mathcal{A}}(0)$.

Figures 1 and 2 illustrate the difference between multi-bump homoclinic orbits and the ones we find in this paper.

2. Extremal homology classes

In this section, we present some basic properties concerning extremal homology classes, they will be used to construct the homoclinic orbits to the given Aubry set. The first of them is the following lemma.

Lemma 2.1. *If $c \in \partial P_0$ and the hypothesis* (H$_3$) *is satisfied, then there is a c-minimal invariant measure μ_c, such that* $\operatorname{supp}(\mu) \subset \widetilde{\mathcal{A}}(c) \setminus \widetilde{\mathcal{A}}(0)$ *and the rotation vector $\rho(\mu_c)$ of this c-minimal measure, as an element of $H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{R})$, is nonzero.*

Proof. For all $c' \in \operatorname{int} P$, the Aubry sets $\widetilde{\mathcal{A}}(c')$ are the same subsets of $TM \times \mathbb{T}$ which are contained in $\widetilde{\mathcal{A}}(c)$ if $c \in \partial P$ (cf. [10]).

Since we have assumed that the 0-minimal measure is uniquely ergodic, $\widetilde{\mathcal{A}}(0)$ has only one static class. For $c \in \partial P_0$, if $\widetilde{\mathcal{A}}(c) \setminus \widetilde{\mathcal{A}}(0) = \emptyset$, then

$$\mathcal{N}(c) = \mathcal{A}(c) = \mathcal{A}(0) = \mathcal{N}(0) \subset U.$$

By the upper semi-continuity of set function $c \to \widetilde{\mathcal{N}}(c)$, there is a constant $\lambda > 1$ such that $\mathcal{N}(\lambda c) \subset U$. It means that c is a interior point of P_0, contradicting to our choice of c.

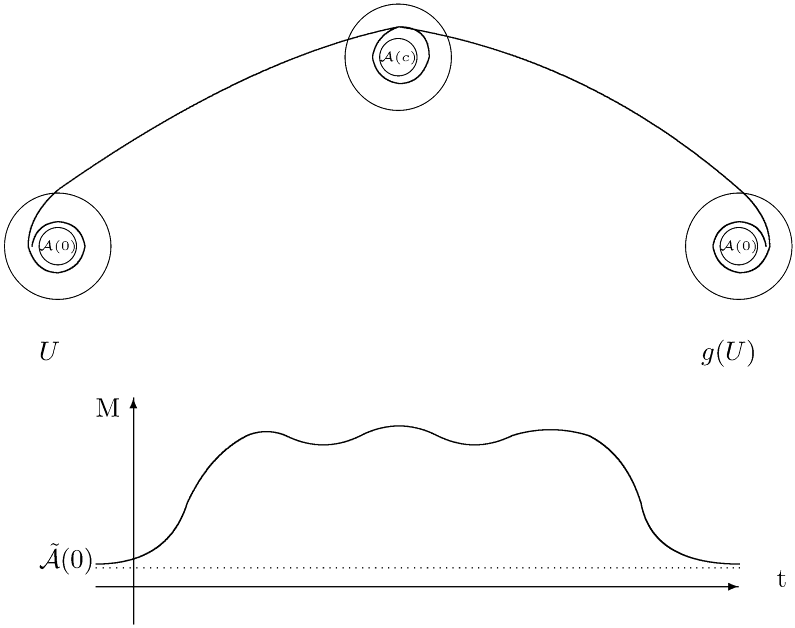

Fig. 2. Our homoclinic orbits.

Suppose $\gamma:\mathbb{R}\to M$ be a c-static orbit such that $(d\gamma(t), t \bmod 1)$ is not in $\widetilde{\mathcal{A}}(0)$. Obviously, its α and ω limit sets must intersect $\mathcal{M}(c)$. If there was no other c-minimal invariant measure contained in $\widetilde{\mathcal{A}}(c)$ except the one in $\widetilde{\mathcal{A}}(0)$, there would be a point $x\in\mathcal{M}(0)|_{t=0}$, two sequences of integers $t_i^l < t_{i+1}^l$, $l=1,2$, $i=1,2,\ldots,$ such that $t_i^l\to+\infty$, $\gamma(-t_i^1)\to x$ and $\gamma(t_i^2)\to x$ as $i\to+\infty$ for $l=1,2$,

$$\Phi_c\big(\gamma\big(-t_i^1\big),\gamma\big(t_j^2\big)\big)=\int\limits_{-t_i^1}^{t_j^2}(L-\eta_c)\big(d\gamma(t),t\big)\,dt\to 0\quad\text{and}$$

$$h_0^\infty\big(x,\gamma\big(-t_i^1\big)\big)+h_0^\infty\big(\gamma\big(t_j^2\big),x\big)\to 0$$

as $i,j\to+\infty$.

We denote the relative homology class of $\gamma|_{[-t_i^1,t_j^2]}$ by $[\gamma_{[-t_i^1,t_j^2]}]$ for sufficiently large i,j. They are well defined since both $\gamma(-t_i^1)$ and $\gamma(t_j^2)$ are in U for such i,j.

If $0\neq[\gamma|_{[-t_i^1,t_j^2]}]\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})$, then

$$\begin{aligned}\delta &\leqslant h_c^\infty\big([\gamma|_{[-t_i^1,t_j^2]}]\big)\leqslant h_0^\infty\big(x,\gamma\big(-t_i^1\big)\big)+\Phi_c\big(\gamma\big(-t_i^1\big),\big(\gamma\big(t_j^2\big)\big)\big)+h_0^\infty\big(\gamma\big(t_j^2\big),x\big)\to 0,\\ &\text{as } i,j\to+\infty.\end{aligned}$$

It is absurd.

If $0=[\gamma|_{[-t_i^1,t_j^2]}]\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})$. As $(d\gamma(t),t)\notin\widetilde{\mathcal{A}}(0)$, we have some positive number $\delta_1>0$ such that

$$\delta_1 \leqslant B_0\big(\gamma(0)\big) \leqslant h_0^\infty\big(x, \gamma\big(-t_i^1\big)\big) + \Phi_0\big(\gamma\big(-t_i^1\big), \gamma\big(t_j^2\big)\big) + h_0^\infty\big(\gamma\big(t_j^2\big), x\big)$$
$$\leqslant h_0^\infty\big(x, \gamma\big(-t_i^1\big)\big) + \Phi_c\big(\gamma\big(-t_i^1\big), \gamma\big(t_j^2\big)\big) + h_0^\infty\big(\gamma\big(t_j^2\big), x\big) \to 0,$$
as $i, j \to +\infty$.

The third inequality follows from the fact $\langle c, [\gamma|_{[-t_i^1, t_j^2]}]\rangle = 0$. It is also absurd.

Therefore, there must exist some new c-minimal measure μ_c in $\widetilde{\mathcal{A}}(c) \setminus \widetilde{\mathcal{A}}(0)$.

If $0 = \rho(\mu_c) \in H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{R})$, then

$$0 = -\alpha(c) = A_c(\mu_c) = \int (L - \eta_c)\, d\mu_c = \int L\, d\mu_c = A_0(\mu_c) \geqslant -\alpha(0) = 0.$$

It means that μ_c is a zero-minimal measure, contradicting the uniqueness hypothesis on zero-minimal measure. So $\rho(\mu_c) \neq 0$. □

Without losing generality, we assume that such c-minimal invariant measure is ergodic. Otherwise we would consider one of its ergodic component whose rotation vector is nonzero.

We denote by $\widetilde{S}$ the static class which contains the new c-minimal invariant measure μ_c and $S = \pi\widetilde{S}$, then:

Lemma 2.2.

$$d\big(\mathcal{A}(0), S\big) \triangleq \min_{x \in \mathcal{A}(0)|_{t=0},\, y \in S|_{t=0}} d_c(x, y) > 0.$$

Proof. It has been proved in [9] that there exists a positive number C such that

$$d_c(\xi, \eta) \leqslant C d(\xi, \eta)^2$$

for $\xi \in \mathcal{A}(c)|_{t=0}$, $\eta \in M$, where d is the metric on M associated to a Riemannian metric.

If $d(\mathcal{A}(0), S) = 0$, there would be two points $\xi \in \mathcal{A}(0)|_{t=0}$ and $\eta \in S|_{t=0}$ such that $d_c(\xi, \eta) = 0$. Since $x \in \mathcal{M}(0)|_{t=0}$ and ξ are in the same static class, we have $d_c(x, \xi) = 0$. Let μ_c be an ergodic minimal measure contained in $\widetilde{S}$ and $y \in \pi \circ \operatorname{supp} \mu_c|_{t=0}$, then $d_c(\eta, y) = 0$. So

$$0 \leqslant d_c(x, y) \leqslant d_c(x, \xi) + d_c(\xi, \eta) + d_c(\eta, y) = 0.$$

According to the definition of d_c, there exist two absolutely continuous curves $\gamma_i : [0, n_i] \to M$, $i = 1, 2$ such that $\gamma_1(0) = \gamma_2(n_2) = x$, $\gamma_1(n_1) = \gamma_2(0) = y$ and small positive number $\delta/4 > \epsilon > 0$ such that

$$\int_0^{n_1} (L - \eta_c)\big(d\gamma_1(t), t\big)\, dt + \int_0^{n_2} (L - \eta_c)\big(d\gamma_2(t), t\big)\, dt \leqslant h_c^\infty(x, y) + h_c^\infty(y, x) + \epsilon$$
$$= d_c(x, y) + \epsilon = \epsilon,$$

where η_c is a closed 1-form satisfying $[\eta_c] = c$.

Since $y \in \pi \circ \operatorname{supp} \mu_c|_{t=0}$, for the ϵ above, there are a sequence of time $\{T_i\}_{i=1}^{+\infty}$ and a sequence of absolutely continuous closed curves $x_i : [0, T_i] \to M$ satisfying $x_i(0) = x_i(T_i) = y$ such that

$$(1) \qquad \int_0^{T_i} (L - \eta_c)\big(dx_i(t), t\big)\, dt \leqslant \epsilon,$$

$$(2) \qquad T_i \to +\infty \quad \text{and} \quad \big\| [x_i|_{[0,T_i]}] \big\| \to +\infty \quad \text{as } i \to +\infty.$$

This follows from the facts that the c-minimal measure μ_c contained in $\widetilde{\mathcal{A}}(c) \setminus \widetilde{\mathcal{A}}(0)$ is ergodic and its rotation vector $\rho(\mu_c)$ is nonzero.

Since $x \in \mathcal{M}(0)|_{t=0}$, there are a sequence of times $\{l_j\}_{j=1}^{+\infty}$ and a sequence of absolutely continuous closed curves $\vartheta_j : [-l_j, 0] \to M$ with $\vartheta_j(-l_j) = \vartheta_j(0) = x$ such that

$$(1) \qquad \int_{-l_j}^{0} (L - \eta_c)\big(d\vartheta_j(t), t\big)\, dt \leqslant \epsilon \quad \text{for all } j, \quad \text{and}$$

$$(2) \qquad \lim_{j \to +\infty} l_j = +\infty \quad \text{and} \quad [\vartheta_j|_{[-l_j,0]}] = 0.$$

This follows from the facts that the 0-minimal measure is ergodic and its rotation vector is zero.

Let us consider the curves $\zeta_j^i : [-l_j, n_1 + n_2 + T_i] \to M$ as follows:

$$\zeta_j^i(t) = \begin{cases} \vartheta_j(t), & t \in [-l_j, 0], \\ \gamma_1(t), & t \in [0, n_1], \\ x_i(t - n_1), & t \in [n_1, n_1 + T_i], \\ \gamma_2(t - n_1 - T_i), & t \in [n_1 + T_i, n_1 + n_2 + T_i]. \end{cases}$$

Take some sufficiently large i, we must have

$$\big[\zeta_j^i\big|_{[-l_j, n_1+n_2+T_i]}\big] = g_i \neq 0 \quad \text{for all } j, \quad \text{and}$$

$$h_c^\infty(g_i) \leqslant \liminf_{j \to +\infty} \int_{-l_j}^{n+m+T_i} (L - \eta_c)\big(d\zeta_j^i(t), t\big)\, dt \leqslant 4\epsilon < \delta.$$

It is against our hypothesis (H_3). $\square$

Since there is a c-minimal invariant measure whose support is outside the neighborhood of $\widetilde{\mathcal{A}}(0)$, and the rotation vector of this measure is nonzero, we have:

Proposition 2.1. *There is a constant $C > 0$, such that*

$$\liminf_{\|g\| \to +\infty} h_c^\infty(g) < C.$$

Proof. Suppose μ_c be an ergodic minimal measure contained in $\widetilde{S}$. Let x be a point in $\mathcal{M}(0)|_{t=0}$ and $y \in \pi \circ \operatorname{supp} \mu_c|_{t=0}$. For any $1 > \epsilon > 0$, there are two sufficiently large integers n_1, n_2 and $\gamma_i : [0, n_i] \to M$, $i = 1, 2$, such that $\gamma_1(0) = \gamma_2(n_2) = x$, $\gamma_1(n_1) = \gamma_2(0) = y$ and

$$\int_0^{n_1} (L - \eta_c)\big(d\gamma_1(t), t\big)\, dt + \int_0^{n_2} (L - \eta_c)\big(d\gamma_2(t), t\big)\, dt \leqslant d_c(x, y) + \epsilon.$$

We take $\zeta_j^i : [-l_j, n_1 + n_2 + T_i] \to M$ as in the previous lemma. For any i, $[\zeta_j^i|_{[-l_j, n_1+n_2+T_i]}]$ is constant for all j and we denote it by g_i.

$$h_c^\infty(g_i) \leqslant \liminf_{j\to+\infty} \int_{-l_j}^{n_1+n_2+T_i} (L - \eta_c)\big(d\zeta_j^i(t), t\big)\, dt \leqslant d_c(x, y) + 3\epsilon < C_1 \;\big(\triangleq d_c(x, y) + 3\big).$$

Obviously, $\|g_i\| \to \infty$ as $i \to \infty$. So, if we take $C = \max_{(x,y)\in M\times M} d_c(x, y) + 3$, then

$$\liminf_{\|g\|\to+\infty} h_c^\infty(g) \leqslant \liminf_{i\to+\infty} h_c^\infty(g_i) < C. \qquad \square$$

We call the sequence $\{g_i\}_{i=1}^{+\infty} \subset H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{Z})$ irreducible minimizing sequence of first homology if

$$\lim_{i\to+\infty} h_c^\infty(g_i) = \liminf_{\|g\|\to+\infty} h_c^\infty(g),$$

we have the following:

Lemma 2.3. *There exists an integer I, such that for all $i > I$,*

$$h_c^\infty(g_i) < h_c^\infty(g') + h_c^\infty(g''), \tag{1}$$

where $g_i = g' + g''$ and both g' and g'' are nonzero.

Proof. First all of, we have

$$h_c^\infty(g) \leqslant h_c^\infty(g') + h_c^\infty(g'') \tag{2}$$

for all $g \in H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{Z})$ and $g = g' + g''$, where g' and g'' are nonzero. This lemma means inequality (2) holds strictly for all $i > I$. In fact, if for some i

$$h_c^\infty(g_i) = \sum_{k=1}^{N_i} h_c^\infty(\theta_k), \tag{3}$$

where $0 \neq \theta_k \in H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{Z})$ and $g_i = \sum_{k=1}^{N_i} \theta_k$. Since

$$\inf_{g\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})} h_c^\infty(g) = \delta > 0,$$

there exists an integer $N = [C/\delta] + 1$ such that all $N_i < N$. This means that if $h_c^\infty(g_i)$ can be read as the sum of some $h_c^\infty(\theta)$ then the number of the summand must be less than N uniformly.

For any large integer I, if there is an $i > I$ such that (3) holds, we can take a new homology class $g_{i'}$ with even larger $\|[g_{i'}]\|$ such that

$$h_c^\infty(g_{i'}) < h_c^\infty(g_i) - \delta.$$

From this, we can get a new sequence of homology classes $\{g_{i'}\}_{i=1}^{+\infty} \subset H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{Z})$ such that

$$\lim_{i \to +\infty} \|g_{i'}\| \to +\infty \quad \text{and}$$
$$\liminf_{i \to +\infty} h_c^\infty(g_{i'}) \leqslant \lim_{i \to +\infty} h_c^\infty(g_i) - \delta = \liminf_{\|g\| \to +\infty} h_c^\infty(g) - \delta.$$

This contradicts our choice of $\{g_i\}_{i=1}^{+\infty}$. $\square$

Remark 2.1. If g satisfies the inequality (1) in Lemma 2.3, we call it an extremal homology class (cf. [4]).

By the lemmas above, we can choose a sequence of extremal homology classes $\{g_i\}_{i=1}^{+\infty}$ such that

$$\lim_{i \to +\infty} h_c^\infty(g_i) = \liminf_{\|g\| \to +\infty} h_c^\infty(g) \leqslant C.$$

We call it an irreducible minimizing sequence. Obviously, for each $g_i \in \{g_i\}_{i=1}^{+\infty}$, it must be an extremal homology class.

3. Construction of homoclinic orbits

In this section, we will construct the homoclinic orbits with respect to each g_i in the irreducible minimizing sequence. The homoclinic orbits also can be constructed using the covering methods introduced independently in [5,7].

Let us define the set of forward and backward c-semi-static curves as follows:

$$\widetilde{\mathcal{N}}^+(c) = \big\{(z, s) \in TM \times \mathbb{T}\colon \pi \circ X(t)\big|_{[s,+\infty)} \text{ is } c\text{-semi-static}\big\},$$
$$\widetilde{\mathcal{N}}^-(c) = \big\{(z, s) \in TM \times \mathbb{T}\colon \pi \circ X(t)\big|_{(-\infty,s]} \text{ is } c\text{-semi-static}\big\}.$$

If $\widetilde{\mathcal{M}}(c)$ is uniquely ergodic for some c, then both $\pi \circ \widetilde{\mathcal{N}}^+(c)$ and $\pi \circ \widetilde{\mathcal{N}}^-(c)$ are equal to $M \times \mathbb{T}$ and $\widetilde{\mathcal{N}}^+(c)|_{\mathcal{A}(c)} = \widetilde{\mathcal{N}}^-(c)|_{\mathcal{A}(c)} = \widetilde{\mathcal{A}}(c)$ (cf. [2] or [6]).

For the construction of homoclinic orbits, we need the following lemma.

Lemma 3.1. *There exists a connected open neighborhood U_1 of $\mathcal{A}(0)$ in U, such that for any $(x, \tau) \in U_1$, both the forward 0-semi-static orbit $\pi \circ X^+(t)|_{[\tau,+\infty)}$ and the backward 0-semi-static $\pi \circ X^-(t)|_{(-\infty,\tau]}$ are in U, where $\pi \circ X^\pm(\tau) = x$.*

Proof. Let us consider the case of the forward 0-semi-static orbits. If Lemma 3.1 is not true, then there exist a sequence of points $(x_i, \tau_i) \to (x, \tau) \in \mathcal{A}(0)$ as $i \to +\infty$, a sequence of vectors v_i such that $(x_i, v_i, \tau_i) \in \widetilde{\mathcal{N}}^+(0)$, a sequence of forward 0-semi-static orbits $\pi \circ X_i^+(t) : [\tau_i, +\infty) \to M \times \mathbb{T}$ such that $X_i^+(\tau_i) = (x_i, v_i, \tau_i)$ and a sequence of times t_i such that $\pi \circ X_i^+(t) \in U$ for $t < t_i$ and $\pi \circ X_i^+(t_i) \in \partial U$. Clearly, there is a constant K such that $\|X_i^+(t)\| \leqslant K$ for all i and time $t \in [\tau_i, +\infty)$, so we have $(x_i, v_i, \tau_i) \to (x, v, \tau)$ and $\pi \circ X(t) : [\tau, +\infty) \to M \times \mathbb{T}$ is a forward 0-semi-static orbit, where $X(t) : [\tau, +\infty) \to TM \times \mathbb{T}$ is the limit of X_i^+ with $X(\tau) = (x, v, \tau)$, by passing to a subsequence if necessary. Obviously $(x, v, \tau) \in \widetilde{\mathcal{A}}(0)$.

There are two cases:

1. If there is a constant T such that $\liminf_{i \to +\infty} t_i < T$. Without losing of generality, we can suppose that $\lim_{i \to +\infty} t_i = T$. By passing to a subsequence if necessary, $X_i^+(t)|_{[\tau_i, t_i]}$ converges to $X(t)|_{[\tau, T]}$ uniformly. By the compactness of ∂U and continuity property of Lagrangian flow, we have $\pi \circ X(T) \in \partial U$. Since (x, v, τ) is in $\widetilde{\mathcal{A}}(0)$ and $\widetilde{\mathcal{A}}(0)$ is invariant for Lagrangian flow, $\mathcal{A}(0) \cap \partial U \neq \emptyset$. This is against the choice of U.

2. If $\lim_{i \to +\infty} t_i = +\infty$, we define $Y_i^+(t) : [\tau_i - [t_i], +\infty) \to TM \times \mathbb{T}$ as following:

$$Y_i^+(t) = X_i^+\big(t + [t_i]\big).$$

Obviously, $\pi \circ Y_i^+$ is forward 0-semi-static orbit. And since $\lim_{i \to +\infty} t_i = +\infty$, there is a curve $Y(t) : \mathbb{R} \to TM \times \mathbb{T}$ such that Y_i^+ converges to Y uniformly on any compact interval $[a, b]$ of $\mathbb{R}$ by passing to a subsequence if necessary. Obviously, $\pi \circ Y : \mathbb{R} \to M \times \mathbb{T}$ is 0-semi-static. By the compactness of ∂U and $\pi \circ Y_i^+(t_i - [t_i]) \in \partial U$, we have $Y_i^+(t_i - [t_i]) \to (x, v, \tau)$ and $(x, \tau) \in \partial U$. So $Y(\tau) = (x, v, \tau)$ and $\pi \circ Y(\tau) = (x, \tau) \in \partial U$. Since the Aubry set $\widetilde{\mathcal{A}}(0)$ has only one static class, then $Y(t) \in \widetilde{\mathcal{A}}(0)$. It means that $\pi \circ Y(\tau) \in \mathcal{A}(0) \cap \partial U$. But this contradicts our choice of U.

For the case of the backward 0-semi-static orbits, the proof is similar and we omit it. $\square$

In the remaining of this section, we construct a homoclinic orbit for each $g_i \in \{g_i\}_{i=1}^{+\infty}$. For this purpose we choose a $g_i \in \{g_i\}_{i=1}^{+\infty}$, fix and denote it by g for brevity. In order to construct the homoclinic orbit with respect to g, we suppose $\gamma_i : [-T_0^i, T_1^i] \to M$ be the minimizer satisfying:

$$A_c(\gamma_i) \triangleq \int_{-T_0^i}^{T_1^i} (L - \eta_c)\big(d\gamma_i(t), t\big)\, dt \leqslant h_c^\infty(g) + \frac{1}{i}$$

for each $i \in \mathbb{N}$, where $\gamma_i(-T_0^i) = \gamma(T_1^i) = x \in \mathcal{M}(0)|_{t=0}$ and $[\gamma_i|_{[-T_0^i, T_1^i]}] = g$. Let

$$\tau_o^i = \sup\big\{t:\ \gamma_i|_{[-T_0^i, t]} \subset U_1\big\}, \qquad \tau_e^i = \inf\big\{t:\ \gamma_i|_{[t, T_1^i]} \subset U_1\big\}.$$

Since the action $A_c(\gamma_i)$ of $\gamma_i(t)$ is uniformly bounded for all $i \in \mathbb{N}$ and the Lagrangian L has fiberwise superlinear growth, there exists a constant K such that

$$\big\|d\gamma_i(t)\big\| \leqslant K$$

for all $i \in \mathbb{N}$ and $t \in [-T_0^i, T_1^i]$.

By the completeness hypothesis of the flow determined by E–L vector field, all γ_i are C^1 curves. Without losing of generality, we can suppose $\tau_o^i \in [0,1)$. Obviously, there is a C^1 curve $\gamma:\mathbb{R}\to M$ which is the minimizer of L, and a subsequence of $\{\gamma_i\}_{i=1}^{\infty}$ which converges to γ uniformly on any compact set I of $\mathbb{R}$. For simplicity of notations, we suppose that the subsequent of $\{\gamma_i\}$ is itself. It is clearly that $(\gamma_i(\tau_o^i),\tau_o^i)\to(\gamma(\tau),\tau)\in\partial U_1$ since $(\gamma_i(\tau_o^i),\tau_o^i)\in\partial U_1$ and ∂U_1 is compact.

A curve $\gamma:\mathbb{R}\to M$ is $\widetilde{M}$-semi-static if its lift to $\widetilde{M}$, denoted by $\tilde{\gamma}:\mathbb{R}\to\widetilde{M}$, satisfies:

$$A_c(\tilde{\gamma}_{[a,b]}) = \inf_{\substack{t=a \bmod 1,\, t'=b \bmod 1\\ t'\geqslant t+1}} \min_{\substack{\xi(t)=\tilde{\gamma}(a),\,\xi(t')=\tilde{\gamma}(b)\\ \xi\in C^1([t,t'],M)}} \int_t^{t'} (L-\eta_c)\big(d\xi(s),s\big)\,ds$$

for any $[a,b]\subset\mathbb{R}$.

Theorem 3.1. *Let $g\in H_1(M\times\mathbb{T},\mathcal{A}(0),\mathbb{Z})$ be an extremal homology class, then there is an $\widetilde{M}$-semi-static orbit $\gamma:\mathbb{R}\to M$ such that $(\gamma(t),\dot{\gamma}(t),t \bmod 1)$ is a homoclinic orbit to $\widetilde{\mathcal{A}}(0)$ and $[\gamma|_{t\in\mathbb{R}}]=g$.*

Proof. Suppose $\tilde{\gamma}_i(t):\mathbb{R}\to\widetilde{M}$ be the lift of γ_i such that $\tilde{\gamma}_i$ converges to $\tilde{\gamma}$ uniformly on any compact set of $\mathbb{R}$ by passing to a subsequence if necessary. If γ is not an $\widetilde{M}$-semi-static orbit, then there exist a time interval $[a,b]$, an integer n and a minimizer $\zeta:[a,b+n]\to\widetilde{M}$ satisfying $\zeta(a)=\tilde{\gamma}(a)$, $\zeta(b+n)=\tilde{\gamma}(b)$ such that

$$A_c(\tilde{\gamma}|_{[a,b]}) > A_c(\zeta|_{[a,b+n]}).$$

Since $\tilde{\gamma}_i|_{[a,b]}$ converges to $\tilde{\gamma}|_{[a,b]}$ uniformly on $[a,b]$ as i tends to infinity. So, for any $\epsilon>0$, there is an integer N_1 such that for all $i>N_1$

$$A_c(\tilde{\gamma}_i|_{[a,b]}) \geqslant A_c(\tilde{\gamma}|_{[a,b]})-\epsilon.$$

By the completeness of Euler–Lagrange flows, we have $\zeta(t)\in C^1([a,b+n],M)$. Let K be a constant such that $\|d\zeta(t)\|\leqslant K$ for all $t\in[a,b+n]$.

On the other hand, $d(\tilde{\gamma}_i(a),\tilde{\gamma}(a))\to 0$, $d(\tilde{\gamma}_i(b),\tilde{\gamma}(b))\to 0$ as $i\to+\infty$. So, for any $\delta>0$, there is an integer N_2 such that $d(\tilde{\gamma}_i(a),\zeta(a))=d(\tilde{\gamma}_i(a),\tilde{\gamma}(a))\leqslant\delta$ and $d(\tilde{\gamma}_i(b),\zeta(b+n))=d(\tilde{\gamma}_i(b),\tilde{\gamma}(b))\leqslant\delta$ for all $i>N_2$. Moreover,

$$d\big(\tilde{\gamma}_i(a),\zeta(a+\delta)\big)\leqslant d\big(\tilde{\gamma}_i(a),\zeta(a)\big)+d\big(\zeta(a),\zeta(a+\delta)\big)\leqslant(K+1)\delta.$$

As the same reason,

$$d\big(\zeta(b+n-\delta),\tilde{\gamma}_i(b)\big)\leqslant(K+1)\delta.$$

Let us consider the geodesic $x_i:[a,a+\delta]\to\widetilde{M}$ connecting $\tilde{\gamma}_i(a)$ and $\zeta(a+\delta)$, and the geodesic $y_i:[b+n-\delta,b+n]\to\widetilde{M}$ connecting $\zeta(b+n-\delta)$ and $\tilde{\gamma}_i(b)$. Obviously, $\|dx_i\|\leqslant(K+1)$, $\|dy_i\|\leqslant(K+1)$. So, there is a constant C such that

$$\left|A_c(x_i|_{[a,a+\delta]})\right| \leqslant C\delta, \qquad \left|A_c(y_i|_{[b+n-\delta,b+n]})\right| \leqslant C\delta,$$
$$\left|A_c(\zeta|_{[a,a+\delta]})\right| \leqslant C\delta, \qquad \left|A_c(\zeta|_{[b+n-\delta,b+n]})\right| \leqslant C\delta.$$

Let us consider the following curves:

$$\tilde{\gamma}_i'(t) = \begin{cases} \tilde{\gamma}_i(t), & t \in [-T_0^i, a], \\ x_i(t), & t \in [a, a+\delta], \\ \zeta(t), & t \in [a+\delta, b+n-\delta], \\ y_i(t), & t \in [b+n-\delta, b+n], \\ \tilde{\gamma}_i(t-n), & t \in [b+n, T_1^i + n]. \end{cases}$$

Let $\Delta = A_c(\tilde{\gamma}|_{[a,b]}) - A_c(\zeta|_{[a,b+n]})$, $\epsilon < \Delta/4$, $\delta < \Delta/16C$, we have

$$\begin{aligned} A_c(\tilde{\gamma}_i) - A_c(\tilde{\gamma}_i') &= \int_{-T_0^i}^{T_1^i} (L-\eta_c)\big(d\tilde{\gamma}_i(t), t\big)\,dt - \int_{-T_0^i}^{T_1^i+n} (L-\eta_c)\big(d\tilde{\gamma}_i'(t), t\big)\,dt \\ &= A_c(\tilde{\gamma}_i|_{[a,b]}) - A_c(\zeta|_{[a,b+n]}) - A_c(x_i|_{[a,a+\delta]}) - A_c(y_i|_{[b+n-\delta,b+n]}) \\ &\quad + A_c(\zeta|_{[a,a+\delta]}) + A_c(\zeta|_{[b+n-\delta,b+n]}) \\ &\geqslant A_c(\tilde{\gamma}|_{[a,b]}) - A_c(\zeta|_{[a,b+n]}) - \epsilon - 4C\delta \geqslant \Delta/2 \end{aligned}$$

for all $i > \max\{N_1, N_2\}$.

Let $Pr : \widetilde{M} \to M$ be the projection. Obviously, $[(Pr \circ \tilde{\gamma}_i')|_{[-T_0^i, T_1^i+n]}] = g$ for all i. As $i \to +\infty$, we will have

$$h_c^\infty(g) = \lim_{i\to+\infty} A_c(\tilde{\gamma}|_{[-T_0^i,T_1^i]}) \geqslant \liminf_{i\to+\infty} A_c(\tilde{\gamma}_i'|_{[-T_0^i,T_1^i+n]}) + \Delta/2 \geqslant h_c^\infty(g) + \Delta/2.$$

It is absurd.

According to Lemma 3.1, $(\gamma(t), \dot{\gamma}(t), t \bmod 1)$ is a homoclinic orbit to $\widetilde{\mathcal{A}}(0)$. In fact, if we suppose $\tau_o^i \in [0,1)$, then there is a constant T such that $\tau_e^i - \tau_o^i \leqslant T$ since g satisfies inequality (1) in Lemma 2.3. So $\gamma_i|_{[\tau_o^i, \tau_e^i]}$ converges uniformly to γ by passing to subsequence if necessary. Since the compactness of ∂U_1 and $(\gamma_i(\tau_o^i), \tau_o^i) \in \partial U_1$, $(\gamma_i(\tau_e^i), \tau_e^i) \in \partial U_1$. Suppose

$$\big(\gamma_i\big(\tau_o^i\big), \tau_o^i\big) \to (x, \tau_o) \in \partial U_1, \qquad \big(\gamma_i\big(\tau_e^i\big), \tau_e^i\big) \to (y, \tau_e) \in \partial U_1.$$

Since $\gamma_i|_{[-T_0^i, \tau_o^i]}$ and $\gamma_i|_{[\tau_e^i, T_1^i]}$ are in U_1, their limits $\gamma|_{(-\infty,\tau_o]}$ and $\gamma|_{[\tau_e,+\infty)}$ are backward and forward 0-semi-static orbit respectively, it follows from Lemma 3.1 that they stay in U. What remains to prove is that both the α-limit set and the ω-limit set of $(\gamma(t), \dot{\gamma}(t), t \bmod 1)$ are in $\widetilde{\mathcal{A}}(0)$. We only prove the case of ω-limit set. For the case of α-limit set the proof is similar and we omit it. As a matter of fact, if (x, v, s) is a point in the ω-limit set of $(\gamma(t), \dot{\gamma}(t), t \bmod 1)$, then there is a sequence of times $\{t_k\}_{k=1}^{+\infty}$ satisfying $t_k = s \bmod 1$ such that $(\gamma(t_k), \dot{\gamma}(t_k), t_k \bmod 1) \to (x, v, s)$ and $t_{k+1} - t_k \to +\infty$ as $k \to +\infty$. Let

$$\zeta_k = \gamma\big(t + [t_k]\big).$$

Taking a subsequence if necessary, we can suppose that there is a C^1 curve $\zeta:\mathbb{R}\to M\times T$ such that ζ_k converges to ζ uniformly on any compact set I of $\mathbb{R}$. Obviously, $(\zeta(s),\dot\zeta(s),s)=(x,v,s)$. If we proved that ζ is a 0-static orbit, then $(x,v,s)\in\widetilde{\mathcal{A}}(0)$ and the theorem has been proved.

In order to prove ζ is a 0-static orbit, we suppose $t'\geqslant t+1$,

$$\begin{aligned}
&A(\zeta|_{[t,t']})+\Phi_0\big((\zeta(t'),t' \bmod 1),(\zeta(t),t \bmod 1)\big)\\
&\quad\leqslant\liminf_{k\to+\infty}A(\gamma|_{[t+[t_k],t'+[t_k]]})+\Phi_0\big(\big(\zeta(t'),t' \bmod 1\big),\big(\zeta(t),t \bmod 1\big)\big)\\
&\quad\leqslant\liminf_{k\to+\infty}\big(A(\gamma|_{[t_{k-1},t_{k+1}]})-A(\gamma|_{[t_{k-1},t+[t_k]]})-A(\gamma|_{[t'+[t_k],t_{k+1}]})\big)\\
&\qquad+\Phi_0\big(\big(\zeta(t'),t' \bmod 1\big),\big(\zeta(t),t \bmod 1\big)\big)\\
&\quad\leqslant\liminf_{k\to+\infty}A(\gamma|_{[t_{k-1},t_{k+1}]})-\big(\Phi_0\big((x,s),\big(\zeta(t),t \bmod 1\big)\big)\\
&\qquad-\Phi_0\big(\big(\zeta(t'),t' \bmod 1\big),\big(\zeta(t),t \bmod 1\big)\big)+\Phi_0\big(\big(\zeta(t'),t' \bmod 1\big),(x,s)\big)\big)\\
&\quad\leqslant\liminf_{k\to+\infty}A(\gamma|_{[t_{k-1},t_{k+1}]})\leqslant 0.
\end{aligned}$$

The last inequality follows from the fact that the sum

$$\sum_{k=1}^{n}A(\gamma|_{[t_{2k-1},t_{2k+1}]})=A(\gamma|_{[t_1,t_{2n+1}]})$$

is bounded, which implies the lim inf is not positive (cf. [2]).

If $0\neq[\gamma|_{t\in\mathbb{R}}]=\theta\neq g$, then we can see that

$$h_c^\infty(g)=h_c^\infty(\theta)+h_c^\infty(g-\theta)$$

and this is against the choice of g, i.e., the conclusion is true.

In fact, we suppose $x\in\mathcal{M}(0)|_{t=0}$. Since $(\gamma(t),\dot\gamma(t),t \bmod 1)$ is the homoclinic orbit to $\widetilde{\mathcal{A}}(0)$, for any $\epsilon>0$, there exist two integers T, I and sufficiently small number $\delta>0$ such that $d(\gamma_i(T),x)<\delta$ and $d_c(x,\gamma_i(T))\leqslant\epsilon$ for any $i>I$. The second inequality follows from the facts that d_c is continuous on $M\times M$ and $d_c(x,x)=0$. So, we can get two sequences of time T_i^l, T_i^r and a sequence of curve $\zeta_i:[0,T_i^l+T_i^r]\to M$ satisfying $\zeta_i(0)=\zeta_i(T_i^l+T_i^r)=\gamma_i(T)$, $\zeta_i(T_i^l)=x$ such that

$$[\zeta_i|_{[0,T_i^l]}]=[\zeta_i|_{[T_i^l,T_i^r]}]=0,\qquad \int_0^{T_i^l+T_i^r}(L-\eta_c)\big(d\zeta_i(t),t\big)\,dt\leqslant 2\epsilon,$$

$$\lim_{i\to+\infty}T_i^l=\lim_{i\to+\infty}T_i^r=+\infty.$$

We construct two curves:

$$\iota_i^l(t)=\begin{cases}\gamma_i(t), & t\in[-T_0^i,T],\\ \zeta_i(t-T), & t\in[T,T+T_i^l],\end{cases}$$

$$\iota_i^r(t) = \begin{cases} \zeta_i(t+T_i^l), & t \in [0, T_i^r], \\ \gamma_i(t-T_i^r+T), & t \in [T_i^r, T_1^i+T_i^r-T]. \end{cases}$$

Obviously,

$$[\iota_i^l|_{[-T_0^i, T+T_i^l]}] = \theta, \qquad [\iota_i^r|_{[0, T_1^i+T_i^r-T]}] = g-\theta.$$

So, we have

$$\begin{aligned} h_c^\infty(g) &\geqslant \liminf_{i\to+\infty}\left(\int_{-T_0^i}^{T+T_i^l}(L-\eta_c)\big(d\iota_i^l(t),t\big)\,dt + \int_0^{T_1^i+T_i^r-T}(L-\eta_c)\big(d\iota_i^r(t),t\big)\,dt\right) - 3\epsilon \\ &\geqslant h_c^\infty(\theta) + h_c^\infty(g-\theta) - 3\epsilon. \end{aligned}$$

From the arbitrary of ϵ, the inequalities above mean:

$$h_c^\infty(g) = h_c^\infty(\theta) + h_c^\infty(g-\theta),$$

where $0 \neq \theta \neq g$. $\square$

Since there are infinitely many extremal homology classes according to the discussion in Section 2, there are infinitely many homoclinic orbits to $\widetilde{\mathcal{A}}(0)$. Obviously these homoclinic orbits are $\widetilde{M}$-minimizers. Up to now, the existence of the homoclinic orbits has been proved. For convenience, we also call $\gamma : \mathbb{R} \to M$ a homoclinic orbit to $\widetilde{\mathcal{A}}(0)$ for brevity if $(\gamma(t), \dot\gamma(t), t \bmod 1)$ is a homoclinic orbit to $\widetilde{\mathcal{A}}(0)$.

4. Some properties of homoclinic orbits

In this section, we show some properties of the homoclinic orbits constructed in the previous section. From these properties, we will see that these homoclinic orbits have very different pictures from so called multi-bump solutions.

Let $\gamma_i : \mathbb{R} \to M$ be the homoclinic orbit to $\widetilde{\mathcal{A}}(0)$ whose relative homology class is g_i, i.e. $[\gamma_i|_{t\in\mathbb{R}}] = g_i$, let

$$\iota_o^i = \sup\{t\colon \gamma_i|_{(-\infty,t]} \subset U_1\}, \qquad \iota_e^i = \inf\{t\colon \gamma_i|_{[t,+\infty)} \subset U_1\}.$$

By the periodicity of Lagrangian L with respect to time, we can assume $\iota_o^i \in [0, 1)$ for all i.

Proposition 4.1.

$$\lim_{i\to+\infty}(\iota_e^i - \iota_o^i) = +\infty.$$

Proof. First of all, for any i, there have two sequences of integer t_j^i and T_j^i with $\lim_{j\to+\infty} t_j^i = +\infty$, $\lim_{j\to+\infty} T_j^i = +\infty$ such that

$$\left(\int_{-t_j^i}^{\iota_o^i} + \int_{\iota_e^i}^{T_j^i}\right)(L-\eta_c)\big(\gamma_i(t),t\big)\,dt \to h_c^\infty\big(\big(\gamma_i\big(\iota_e^i\big),\big(\iota_e^i \bmod 1\big)\big),\big(\gamma_i\big(\iota_o^i\big),\iota_o^i\big)\big).$$

As a matter of fact, take $x \in \mathcal{M}(0)|_{t=0}$, there exist two sequences of time t_j^i, T_j^i, $j = 1, 2, \ldots,$ such that

$$d\big(\gamma_i\big(-t_j^i\big),x\big) \to 0, \qquad d\big(\gamma_i\big(T_j^i\big),x\big) \to 0 \quad \text{and}$$
$$t_j^i \to +\infty, \qquad T_j^i \to +\infty$$

as $j \to +\infty$. Since $\gamma_i|_{(-\infty,\iota_o^i]}$ is c-semi-static and h_c^∞ is continuous, we have

$$h_c^\infty\big((x,0),\big(\gamma_i\big(\iota_o^i\big),\iota_o^i\big)\big) = \lim_{j\to+\infty}\int_{-t_j^i}^{\iota_o^i}(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt.$$

As the same reason

$$h_c^\infty\big(\big(\gamma_i\big(\iota_e^i\big),\big(\iota_e^i \bmod 1\big)\big),(x,0)\big) = \lim_{j\to+\infty}\int_{\iota_e^i}^{T_j^i}(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt.$$

Since $x \in \mathcal{M}(0)|_{t=0}$ and the choice of η_c, we have

$$\begin{aligned} &h_c^\infty\big(\big(\gamma_i\big(\iota_e^i\big),\big(\iota_e^i \bmod 1\big)\big),\big(\gamma_i\big(\iota_o^i\big),\iota_o^i\big)\big) \\ &\quad = h_c^\infty\big(\big(\gamma_i\big(\iota_e^i\big),\big(\iota_e^i \bmod 1\big)\big),(x,0)\big) + h_c^\infty\big((x,0),\big(\gamma_i\big(\iota_o^i\big),\iota_o^i\big)\big). \end{aligned}$$

Obviously,

$$\lim_{j\to+\infty}\int_{-t_j^i}^{T_j^i}(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt = h_c^\infty(g_i),$$

for all i. Then there exists a constant C such that

$$\left|\lim_{j\to+\infty}\int_{-t_j^i}^{T_j^i}(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt\right| \leqslant C. \tag{4}$$

So, by the compactness of ∂U and the continuity of h_c^∞, there are two constants B_0, B_1 such that

$$B_0 \leqslant \int_{\iota_o^i}^{\iota_e^i}(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt \leqslant B_1 \quad \text{for all } i.$$

Clearly, there is a constant K such that $\|d\gamma_i(t)\| \leqslant K$ for all i and $t \in \mathbb{R}$. Since we have assumed L has the fiberwise superlinear growth, there are two constants K_0 and $K_1 > 0$ such that

$$K_0\big(\iota_e^i - \iota_o^i\big) \leqslant \int_{\iota_o^i}^{\iota_e^i} L\big(d\gamma_i(t), t\big)\, dt \leqslant K_1\big(\iota_e^i - \iota_o^i\big).$$

It results from these two inequalities that

$$K_0\big(\iota_e^i - \iota_o^i\big) - B_1 \leqslant \langle c, g_i \rangle \leqslant K_1\big(\iota_e^i - \iota_o^i\big) - B_0.$$

As there are at most finite many g_i such that $\langle c, g_i \rangle = 0$, and $\|g_i\| \to +\infty$ as $i \to +\infty$, we have

$$\lim_{i \to +\infty} \big(\iota_e^i - \iota_o^i\big) = +\infty. \qquad \square$$

Let us consider the probability measure μ_i evenly distributed along $\gamma_i|_{[\iota_o^i, \iota_e^i]}$. Suppose μ_c be a vague limit of μ_i.

Theorem 4.1. *μ_c is a c-minimal invariant measure and as the element of $H_1(M \times \mathbb{T}, \mathcal{A}(0), \mathbb{R})$, $\rho(\mu_c) \neq 0$.*

Proof. We can assume that μ_c is an ergodic measure, otherwise let us consider an ergodic component of μ_c.

First of all, there is a constant $K > 0$, for any s, s' satisfying $s' \geqslant s + 1$ and all i, such that

$$\int_s^{s'} (L - \eta_c)\big(d\gamma_i(t), t\big)\, dt \leqslant K. \tag{5}$$

In fact, on one hand, we have inequality (4) holding for all i. On the other hand, there is a constant C_1 with the following property [2]: for all $(m, m') \in M \times M$ and s, s' satisfying $s' \geqslant s + 1$, the inequality

$$F_{s,s'}(m, m') = \min_{\gamma \in \Gamma} \int_s^{s'} (L - \eta_c)\big(d\gamma(t), t\big)\, dt \leqslant C_1$$

holds, where

$$\Gamma = \big\{\gamma \in C^1\big([s, s'], M\big):\ \gamma(s) = m,\ \gamma(s') = m'\big\}.$$

If our claim (5) was not true when $K = C + C_1 + 1$, we would have

$$\left(\int_{-t_j^i}^{s} + \int_{s'}^{T_j^i}\right)(L - \eta_c)\big(d\gamma_i(t), t\big)\, dt \leqslant -(C_1 + 1).$$

Let $\xi_i(t):[s,s']\to M$ be the minimizer connecting $\gamma_i(s)$ and $\gamma_i(s')$. Considering the following curve $\tilde{\gamma}_i(t):\mathbb{R}\to M$:

$$\tilde{\gamma}_i(t)=\begin{cases}\gamma_i(t), & t\in[-t_j^i,s],\\ \xi_i(t), & t\in[s,s'],\\ \gamma_i(t), & t\in[s',T_j^i],\end{cases}$$

we have

$$\begin{aligned}\int\limits_{-t_j^i}^{T_j^i}(L-\eta_c)\big(d\tilde{\gamma}_i(t),t\big)\,dt&=\left(\int\limits_{-t_j^i}^{s}+\int\limits_{s'}^{T_j^i}\right)(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt+\int\limits_{s}^{s'}(L-\eta_c)\big(d\xi_i(t),t\big)\,dt\\&\leqslant-(C_1+1)+C_1=-1.\end{aligned}$$

But since $\lim_{j\to+\infty}\gamma_i(-t_j^i)=\lim_{j\to+\infty}\gamma_i(T_j^i)=x\in\mathcal{M}(0)$, we have

$$\lim_{j\to+\infty}\int\limits_{-t_j^i}^{T_j^i}(L-\eta_c)\big(d\tilde{\gamma}_i(t),t\big)\,dt\geqslant h_c^{\infty}(x,x)=0.$$

This contradiction verifies our claim (5).

Let $\xi\in\operatorname{supp}\mu_c|_{t=0}$ and $X(t):\mathbb{R}\to TM\times\mathbb{T}$ the orbit passing $(\xi,0)$ at $t=0$. Clearly, there exist an integer N and a sequence of integer times $\{T_i\}_{i=1}^{+\infty}$, such that $(d\gamma_i(t)|_{[T_i,T_i+s]},t-T_i)$ converges to $X(t)|_{[0,s]}$ uniformly for all $i>N$ and

$$\int\limits_0^s(L-\eta_c)\big(X(t)\big)\,dt\leqslant\int\limits_{T_i}^{T_i+s}(L-\eta_c)\big(d\gamma_i(t),t\big)\,dt+1\leqslant K+1$$

for $s>1$. Since μ_c is an ergodic minimal invariant measure, we have [8]

$$A_c(\mu)=\int(L-\eta_c)\,d\mu=\lim_{s\to+\infty}\frac{1}{s}\int\limits_0^s(L-\eta_c)\big(X(t)\big)\,dt.$$

So, we have

$$A_c(\mu_c)=\lim_{s\to+\infty}\frac{1}{s}\int\limits_0^s(L-\eta_c)\big(X(t)\big)\,dt=0.$$

It implies that μ_c is a c-minimal invariant measure.

$\rho(\mu_c)\neq0$ follows from the choice of $\{g_i\}_{i=1}^{+\infty}$. In fact, if $\rho(\mu_c)=0$, then it is also the 0-minimal invariant measure. From our hypotheses (H_1), it must be contained in U_1. It is against the choice of $\{g_i\}_{i=1}^{+\infty}$. $\square$

The Proposition 4.1 and the Theorem 4.1 illustrate the main feature of the homoclinic orbits constructed in this paper. According to the Proposition 4.1, such homoclinic orbits do not visit a small neighborhood of $\widetilde{\mathcal{A}}(0)$ for many times. According to the Theorem 4.1, the sequence of such homoclinic orbits has a limit c-minimal measure which appears only when the cohomology class c is on the boundary of P.

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